

Non spherical sources of strong gravitational fields out of hydrostatic equilibrium

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Abstract

We describe the departure from equilibrium of matter distributions representing sources for a class of Weyl metric. It is shown that, for extremely high gravitational fields, slight deviations from spherical symmetry may enhance the stability of the system weakening thereby its tendency to a catastrophic collapse. For critical values of surface gravitational potential, in contrast with the exactly spherically symmetric case, the speed of entering the collapse regime decreases substantially, at least for specific cases.

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1 Introduction

“L’INTERVIEWER. - mais croyez-vous que vous devez être d’accord avec ce que la plupart de gens autour de vous pensent?”

“RUTH. - eh bien, c’est-à-dire que, quand je ne le suis pas, je me retrouve toujours à l’hôpital...”

R. D. Laing, A. Esterson, L’équilibre mental, la folie et la famille, Ed. Maspero (Paris, 1971).

As it is well known, since the early seminal work of Israel [1], the only static and asymptotically-flat vacuum space-time possessing a regular horizon is the Schwarzschild solution. For all the others Weyl exterior solutions [2], the physical components of the Riemann tensor exhibit singularities at $r = 2m$.

Since these components represent the true observables of the theory as it appears from the definition of tidal forces [3], it is intuitively clear that for high gravitational fields, the evolution of sources of Weyl space-time should drastically differ from the evolution of spherical sources [4]. It is important to keep in mind that the sharp difference in the behaviour of both types of sources (for very high gravitational fields) will exist independently on the magnitude of multipole moments (higher than monopole) of the Weyl source. This is so because, as the source approaches the horizon, any finite perturbation of the Schwarzschild space-time becomes fundamentally different from any Weyl solution, even when the latter is characterized by parameters whose values are arbitrarily close to those corresponding to the spherical symmetry. This point has been stressed long time ago [5], but usually it has been overlooked.

It is the purpose of this work to study the evolution of axisymmetric sources for very high gravitational fields. This will allow us to put in evidence the role played by the non sphericity (however small) of the source, on the outcome of evolution. However, instead of following the evolution of the system long time after its departure from equilibrium, what would require the use of numerical procedures, we shall evaluate the source immediately after such departure. Here “immediately” means on a time scale smaller than hydrostatic time scale -see section 4 for more details. In doing so we shall avoid the introduction of numerical procedures. On the other hand,

however, we shall obtain only indications about the tendency of the object and not a complete description of its evolution. In spite of this limitation, this approach has proven to be useful in the study of spherically symmetric case (see [6] and [7] and references therein).

As initial configurations, we shall consider two interior metrics. These ones were found some years ago by Stewart *et al.* [8], following a prescription given by Hernández [9] allowing to obtain interior solutions of Weyl space time, from known spherically symmetric interior solutions.

The configurations to be considered are sources of the so-called gamma metric (γ -metric) [10], [11]. This metric, which is also known as Zipoy-Vorhees metric [12], belongs to the family of Weyl's solutions, and is continuously linked to the Schwarzschild space-time through one of its parameters. The motivation for this choice is twofold. On one hand the exterior γ -metric corresponds to a solution of the Laplace equation (in cylindrical coordinates) with the same singularity structure as the Schwarzschild solution (a line segment [10]). In this sense the γ -metric appears as the "natural" generalization of Schwarzschild space-time to the axisymmetric case. On the other hand, the two interior solutions considered have reasonable physical properties and generalize important and useful sources of the Schwarzschild space-time, namely the interior Schwarzschild solution (homogeneous density) and the Adler solution [13].

From all the comments above, it is not difficult to infer the astrophysical relevance of the results here presented. Indeed, spherical symmetry is a common assumption in the study of compact self-gravitating objects (white dwarfs, neutron stars, black holes), furthermore in the specific case of non-rotating black holes, spherical symmetry should be "absolute", according to Israel theorem. Therefore it is pertinent to ask, how do small deviations from this assumption, related to any kind of perturbation (e.g. fluctuations of the stellar matter, external perturbations, etc), affect the dynamics of the system?. The result obtained here, which deserves to be emphasized, is that slight deviations from spherical symmetry seriously modify the departure from equilibrium in the two examples presented. On the other hand, we are well aware of the fact that the γ metric is not the only possible description for the exterior of a compact object and, of course, the two equations of state considered here do not exhaust the list of possible candidates for the equation of state of the stellar matter. However, in view of the properties of the γ metric and the two equations of state considered here, mentioned

above, it is fair to say that the case for the relevance of small deviations from spherical symmetry in the dynamics of compact objects has been established. This means that any conclusion on the structure and evolution of a compact object, derived on the assumption of spherical symmetry should be carefully checked against deviations from that assumption.

This paper is organized as follows. In the next section we shall specify the space-time inside and outside the matter distribution, and the conventions used. In section 3 we give the energy momentum tensor components in terms of variables measured by a locally Minkowskian and comoving observer. The departure from equilibrium is analyzed in section 4, and results are discussed in the last section. We have included an appendix containing the components of the Einstein tensor, the Einstein equations in the limit of small non sphericity and the components of the conservation law.

2 The Space-time

2.1 The exterior space-time

As has been mentioned above, our initial matter configuration is the source of an axially symmetric and static space-time (γ -metric). In cylindrical coordinates, static axisymmetric solutions to Einstein equations are given by the Weyl metric [2]

$$ds^2 = e^{2\lambda} dt^2 - e^{-2\lambda} \left[e^{2\mu} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right], \quad (1)$$

with

$$\lambda_{,\rho\rho} + \rho^{-1} \lambda_{,\rho} + \lambda_{,zz} = 0 \quad (2)$$

and

$$\mu_{,\rho} = \rho \left(\lambda_{,\rho}^2 - \lambda_{,z}^2 \right) \quad \mu_{,z} = 2\rho \lambda_{,\rho} \lambda_{,z}. \quad (3)$$

Observe that (2) is just the Laplace equation for λ (in the Euclidean space).

The γ -metric is defined by [10]

$$\lambda = \frac{\gamma}{2} \ln \left[\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} \right], \quad (4)$$

$$e^{2\mu} = \left[\frac{(R_1 + R_2 + 2m)(R_1 + R_2 - 2m)}{4R_1 R_2} \right]^{\gamma^2}, \quad (5)$$

where

$$R_1^2 = \rho^2 + (z - m)^2 \quad R_2^2 = \rho^2 + (z + m)^2. \quad (6)$$

It is worth noticing that λ , as given by (4), corresponds to the Newtonian potential of a line segment of mass density $\gamma/2$ and length $2m$, symmetrically distributed along the z axis. The particular case $\gamma = 1$, corresponds to the Schwarzschild metric.

It will be useful to work in Erez-Rosen coordinates [12], given by

$$\rho^2 = (r^2 - 2mr) \sin^2 \theta \quad z = (r - m) \cos \theta, \quad (7)$$

which yields the line element as [10]

$$ds^2 = F dt^2 - F^{-1} \left\{ G dr^2 + H d\theta^2 + (r^2 - 2mr) \sin^2 \theta d\varphi^2 \right\}, \quad (8)$$

where

$$F = \left(1 - \frac{2m}{r} \right)^\gamma, \quad (9)$$

$$G = \left(\frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2 \theta} \right)^{\gamma^2 - 1}, \quad (10)$$

and

$$H = \frac{(r^2 - 2mr)^{\gamma^2}}{(r^2 - 2mr + m^2 \sin^2 \theta)^{\gamma^2 - 1}} \quad (11)$$

Now, it is easy to check that $\gamma = 1$ corresponds to the Schwarzschild metric.

The total mass of the source is [10, 11] $M = \gamma m$, and the quadrupole moment is given by

$$Q = \frac{\gamma}{3} m^3 (1 - \gamma^2). \quad (12)$$

So that $\gamma > 1$ ($\gamma < 1$) corresponds to an oblate (prolate) spheroid.

2.2 The interior space-time

The metric within the matter distribution bounded by the surface

$$r = a \quad (13)$$

is given by

$$\begin{aligned}
g_{tt} &= f^{2\gamma} \\
g_{rr} &= -f^{2(1-\gamma)} \Delta^{\gamma^2-2} \Sigma^{1-\gamma^2} \\
g_{\theta\theta} &= -r^2 f^{2\gamma(\gamma-1)} \Phi^{1-\gamma^2} \\
g_{\varphi\varphi} &= -r^2 f^{2(1-\gamma)} \sin^2 \theta
\end{aligned} \tag{14}$$

where f , Δ , Σ and Φ are functions whose specific form depends on the model under consideration.

The two cases to be considered here are the solutions reported in [8], namely

1. The modified constant density Schwarzschild solution

$$f(r) = \frac{3}{2} \sqrt{1 - \frac{a^2}{B^2}} - \frac{1}{2} \sqrt{1 - \frac{r^2}{B^2}} \tag{15}$$

$$\Delta(r) = 1 - \frac{r^2}{B^2} \tag{16}$$

$$\Sigma(r, \theta) = 1 - \frac{r^2}{B^2} + \frac{r^4}{4B^4} \sin^2 \theta \tag{17}$$

$$\Phi(r, \theta) = f^2 + \frac{r^4}{4B^4} V(r) \sin^2 \theta \tag{18}$$

with

$$V(r) = 1 + \frac{6}{a} (a - r), \tag{19}$$

and

$$B^2 = \frac{3}{8\pi\rho_{ss}}, \tag{20}$$

where ρ_{ss} denotes the energy density in the spherically symmetric limit ($\gamma = 1$)

2. The modified Adler solution

$$f(r) = A + Br^2 \tag{21}$$

$$\Delta(r) = 1 + \frac{Cr^2}{(A + 3Br^2)^{2/3}} \quad (22)$$

$$\Sigma(r, \theta) = 1 + \frac{Cr^2}{(A + 3Br^2)^{2/3}} + \frac{C^2 r^4}{4(A + 3Br^2)^{4/3}} \sin^2 \theta \quad (23)$$

$$\Phi(r, \theta) = (A + Br^2)^2 + \frac{C^2 r^4 V(r)}{4(A + 3Br^2)^{4/3}} \sin^2 \theta \quad (24)$$

$$(25)$$

with

$$V(r) = 1 + \frac{6}{a} \left(1 - \frac{5m}{3a}\right) \left(1 - \frac{m}{a}\right)^{-1} (a - r) \quad (26)$$

and

$$\begin{aligned} A &= \frac{1 - \frac{5m}{2a}}{\left(1 - \frac{2m}{a}\right)^{1/2}} \\ B &= \frac{m}{2a^3 \left(1 - \frac{2m}{a}\right)^{1/2}} \\ C &= -\frac{2m \left(1 - \frac{m}{a}\right)^{2/3}}{a^3 \left(1 - \frac{2m}{a}\right)^{1/3}} \end{aligned} \quad (27)$$

Before closing this section, two remarks are in order:

1. Since we are considering the source described in (14) as an initial state, the time derivatives of functions f , Δ , Σ and Φ will be in principle different from zero.
2. Junction (Darmois) conditions are satisfied at the boundary $r = a$ -see [8] for details.

3 The energy momentum tensor

In order to give physical meaning to the components of the energy momentum tensor in coordinates (t, r, θ, φ) , we shall develop a procedure similar to that

used by Bondi [15] in his study of non static spherically symmetric sources. Thus, we introduce purely local Minkowski coordinates (τ, x, y, z) defined by

$$d\tau = f^\gamma dt \quad (28)$$

$$dx = f^{1-\gamma} \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2} dr \quad (29)$$

$$dy = r f^{\gamma(\gamma-1)} \Phi^{(1-\gamma^2)/2} d\theta \quad (30)$$

$$dz = r \sin(\theta) f^{1-\gamma} d\varphi. \quad (31)$$

Next, since we are assuming that our source does not dissipate energy, then the covariant components of the energy momentum tensor, as measured by a local Minkowskian and comoving with the fluid observer, will be

$$\hat{T}_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_{xx} & p_{xy} & 0 \\ 0 & p_{yx} & p_{yy} & 0 \\ 0 & 0 & 0 & p_{zz} \end{pmatrix}, \quad (32)$$

where ρ is the energy density and in general $p_{xx} \neq p_{yy} \neq p_{zz}$ and $p_{xy} = p_{yx}$. We may write (32) in the form

$$\hat{T}_{\mu\nu} = (\rho + p_{zz}) \hat{U}_\mu \hat{U}_\nu - p_{zz} \eta_{\mu\nu} + (p_{xx} - p_{zz}) \hat{k}_\mu \hat{k}_\nu + (p_{yy} - p_{zz}) \hat{l}_\mu \hat{l}_\nu + 2p_{xy} \hat{k}_{(\mu} \hat{l}_{\nu)}, \quad (33)$$

where $\eta_{\mu\nu}$ denotes the flat space-time metric and

$$\hat{U}_\mu = (1, 0, 0, 0) \quad (34)$$

$$\hat{k}_\mu = (0, 1, 0, 0) \quad (35)$$

$$\hat{l}_\mu = (0, 0, 1, 0) \quad (36)$$

The components of the energy-momentum tensor $T_{\mu\nu}$ in (t, r, θ, φ) coordinates are linked to (33) by

$$T_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta L_\alpha^\gamma L_\beta^\delta \hat{T}_{\gamma\delta}, \quad (37)$$

where $\Lambda_\mu^\nu = \partial x^\nu / \partial x^\mu$ is the local coordinate transformation matrix between Minkowskian coordinates and (t, r, θ, φ) coordinates and the Lorentz matrices L_μ^ν are given by

$$L_t^t = \Gamma \quad L_i^t = L_t^i = -\Gamma w_i \quad L_j^i = L_i^j = \delta_j^i + \frac{(\Gamma - 1)w_i w_j}{w^2}, \quad (38)$$

where

$$w^2 = w_x^2 + w_y^2 \quad \Gamma = \frac{1}{\sqrt{1 - w^2}}, \quad (39)$$

and w_x and w_y denote, respectively, the velocity of a fluid element along the x and y (r and θ) directions, as measured by our local Minkowskian observer as defined by (28)–(31). Observe that we are considering the case $w_z = 0$, which means that the system preserves the reflection symmetry (besides the axial symmetry).

The non vanishing components of Λ_μ^ν are

$$\Lambda_t^\tau = f^\gamma \quad (40)$$

$$\Lambda_r^x = f^{1-\gamma} \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2} \quad (41)$$

$$\Lambda_\theta^y = r f^{\gamma(\gamma-1)} \Phi^{(1-\gamma^2)/2} \quad (42)$$

$$\Lambda_\varphi^z = r \sin(\theta) f^{1-\gamma}. \quad (43)$$

Then, (37) readily gives

$$T_{tt} = f^{2\gamma} \Gamma^2 \left(\rho + p_{xx} w_x^2 + p_{yy} w_y^2 + 2p_{xy} w_x w_y \right) \quad (44)$$

$$T_{tr} = -f \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2} \Gamma \times \\ (\Gamma w_x \rho + p_{xx} w_x \Lambda_x + p_{yy} w_y \Lambda_y + p_{xy} [w_x \Lambda + w_y \Lambda_x]) \quad (45)$$

$$T_{t\theta} = -r f^{\gamma^2} \Phi^{(1-\gamma^2)/2} \Gamma \times \\ (\Gamma w_y \rho + p_{xx} w_x \Lambda + p_{yy} w_y \Lambda_y + p_{xy} [w_y \Lambda + w_x \Lambda_y]) \quad (46)$$

$$T_{rr} = f^{2-2\gamma} \Delta^{\gamma^2-2} \Sigma^{1-\gamma^2} \times \left(\Gamma^2 w_x^2 \rho + p_{xx} \Lambda_x^2 + p_{yy} \Lambda^2 + 2p_{xy} \Lambda \Lambda_x \right) \quad (47)$$

$$T_{r\theta} = r f^{(\gamma-1)^2} \Delta^{-1+\gamma^2/2} \Phi^{(1-\gamma^2)/2} \Sigma^{(1-\gamma^2)/2} \times \left(\Gamma^2 w_x w_y \rho + \Lambda [p_{xx} \Lambda_x + p_{yy} \Lambda_y] + p_{xy} [\Lambda^2 + \Lambda_x \Lambda_y] \right) \quad (48)$$

$$T_{\theta\theta} = r^2 f^{2\gamma(\gamma-1)} \Phi^{1-\gamma^2} \times \left(\Gamma^2 w_y^2 \rho + p_{xx} \Lambda^2 + p_{yy} \Lambda_y^2 + 2p_{xy} \Lambda \Lambda_y \right) \quad (49)$$

$$T_{\varphi\varphi} = r^2 f^{2(1-\gamma)} \sin^2(\theta) p_{zz}, \quad (50)$$

with

$$\Lambda \equiv \frac{(\Gamma - 1) w_x w_y}{w^2}, \quad (51)$$

$$\Lambda_x \equiv 1 + \frac{(\Gamma - 1) w_x^2}{w^2}, \quad (52)$$

$$\Lambda_y \equiv 1 + \frac{(\Gamma - 1) w_y^2}{w^2}, \quad (53)$$

So that

$$T_{\mu\nu} = (\rho + p_{zz}) U_\mu U_\nu - p_{zz} g_{\mu\nu} + (p_{xx} - p_{zz}) k_\mu k_\nu + (p_{yy} - p_{zz}) l_\mu l_\nu + 2p_{xy} k_{(\mu} l_{\nu)}, \quad (54)$$

where U_μ , k_μ and l_μ are obtained after applying the boost velocity (38) and the coordinate transformation (40)–(43) to (34)–(36),

$$U_\mu = \Gamma \left(f^\gamma, -w_x f^{1-\gamma} \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2}, -w_y r f^{\gamma(\gamma-1)} \Phi^{(1-\gamma^2)/2}, 0 \right), \quad (55)$$

$$k_\mu = \left(-\Gamma w_x f^\gamma, f^{1-\gamma} \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2} \Lambda_x, r f^{\gamma(\gamma-1)} \Phi^{(1-\gamma^2)/2} \Lambda, 0 \right), \quad (56)$$

$$l_\mu = \left(-\Gamma w_y f^\gamma, f^{1-\gamma} \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2} \Lambda, r f^{\gamma(\gamma-1)} \Phi^{(1-\gamma^2)/2} \Lambda_y, 0 \right). \quad (57)$$

4 Departure from equilibrium

Let us consider a static axially symmetric source defined by (14), which once submitted to perturbations, departs from equilibrium without dissipation. We shall evaluate the system after such departure, on a time scale such that w_x and w_y remain vanishingly small, whereas their time derivatives though small, will be different from zero.

Thus, just after leaving the equilibrium, the following conditions hold

$$w_x = w_y = w_{x,i} = w_{y,i} \simeq 0, \quad (i = r, \theta, \varphi) \quad (58)$$

$$w_{x,t}, w_{y,t} \neq 0 \quad (\text{small}) \quad (59)$$

From now on, unless otherwise stated, all equations are evaluated at the moment the system starts to deviate from equilibrium.

Then from (44)–(50), we obtain using (58)

$$T_{t\theta} = T_{tr} = 0 \quad (60)$$

which implies, because of (74) and (76)

$$\Delta_{,t} = f_{,t} = \Sigma_{,t} = \Phi_{,t} = 0 \quad (61)$$

where for simplicity we write 0 for $\mathcal{O}(\omega)$ (as we shall do hereafter).

Obviously, spatial derivatives of the above quantities will be also vanishingly small on the time scale under consideration.

Next, we shall evaluate the conservation law $T_{\nu;\mu}^\mu = 0$.

The t -component yields

$$\rho_{,t} = 0 \quad (62)$$

whereas the r and θ component lead, after inspection of (84) and (85), to

$$\begin{aligned} & - f^{-1} f^{-2\varepsilon} \Delta^{-1/2} \Delta^{(\varepsilon+\varepsilon^2/2)} \Sigma^{-(\varepsilon+\varepsilon^2/2)} \{ \omega_{x,t} (\rho + p_{xx}) + \omega_{y,t} p_{xy} \} \\ & - \left[\frac{f_{,r}}{f} (1 + \varepsilon + \varepsilon^2) + \frac{\Phi_{,r}}{2\Phi} (-2\varepsilon - \varepsilon^2) + \frac{2}{r} \right] p_{xx} - p_{xx,r} \\ & - r^{-1} f^{-(2\varepsilon+\varepsilon^2)} \Delta^{-1/2} \Delta^{(\varepsilon+\varepsilon^2/2)} \Phi^{(\varepsilon+\varepsilon^2/2)} \Sigma^{-(\varepsilon+\varepsilon^2/2)} \\ & \times \left\{ \left[\frac{\Sigma_{,\theta}}{\Sigma} (-2\varepsilon - \varepsilon^2) + \cot \theta \right] p_{xy} + p_{xy,\theta} \right\} \end{aligned}$$

$$\begin{aligned}
& - \frac{f_{,r}}{f}(1+\varepsilon)\rho + \left[\frac{1}{r} + \frac{f_{,r}}{f}(-\varepsilon) \right] p_{zz} \\
& + \left[\frac{1}{r} + \frac{f_{,r}}{f}(\varepsilon + \varepsilon^2) + \frac{\Phi_{,r}}{2\Phi}(-2\varepsilon - \varepsilon^2) \right] p_{yy} = 0
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
& - r f^{-1} f^{\varepsilon^2} \Phi^{(-\varepsilon - \varepsilon^2/2)} \{ \omega_{y,t}(\rho + p_{yy}) + \omega_{x,t} p_{xy} \} \\
& - r f^{(2\varepsilon + \varepsilon^2)} \Delta^{1/2} \Delta^{-(\varepsilon + \varepsilon^2/2)} \Phi^{-(\varepsilon + \varepsilon^2/2)} \Sigma^{(\varepsilon + \varepsilon^2/2)} \\
& \times \left\{ \left[\frac{f_{,r}}{f}(1 + 2\varepsilon + 2\varepsilon^2) + \frac{\Phi_{,r}}{\Phi}(-2\varepsilon - \varepsilon^2) + \frac{3}{r} \right] p_{xy} + p_{xy,r} \right\} \\
& - \left[\frac{\Sigma_{,\theta}}{2\Sigma}(-2\varepsilon - \varepsilon^2) + \cot \theta \right] p_{yy} \\
& + \cot \theta p_{zz} - p_{yy,\theta} + \frac{\Sigma_{,\theta}}{\Sigma}(-2\varepsilon - \varepsilon^2) p_{xx} = 0
\end{aligned} \tag{64}$$

where we have assumed

$$\gamma = 1 + \varepsilon$$

and ε may be either positive or negative.

Before going further into the analysis of the equations above it is quite instructive to consider the spherically symmetric situation. In this case we have

$$\Sigma_{,\theta} = \omega_{y,t} = p_{xy} = p_{zz} - p_{yy} = p_{yy,\theta} = \varepsilon = 0 \tag{65}$$

Then, (64) becomes an identity and (63) reads

$$-\omega_{x,t}(\rho + p_{xx}) = \left[p_{xx,r} + \frac{2}{r}(p_{xx} - p_{yy}) + \frac{\nu_{,r}}{2}(\rho + p_{xx}) \right] e^{(\nu - \delta)/2} \tag{66}$$

where

$$\nu_{,r} = 2 \frac{m + 4\pi r^3 p_{xx}}{r(r - 2m)} \tag{67}$$

$$f = e^{\nu/2} \quad ; \quad \Delta \equiv e^{-\delta} = 1 - \frac{2m}{r} \tag{68}$$

with

$$m = \int_0^r 4\pi r^2 \rho dr \tag{69}$$

The physical meaning of (66) is quite transparent. It has the “Newtonian” form

$$Force = mass \times acceleration$$

Indeed, the left-hand side consists of two factors. The time derivative of radial velocity and the inertial mass density. On the right-hand side, we have three possible sources of forces: The pressure gradient (negative), the possible anisotropic pressure contribution and the “gravitational” term (positive), multiplied by the relativistic correction factor $e^{(\nu-\delta)/2}$. As the source becomes more and more compact the relativistic correction factor decreases as $\approx r - 2m$, whereas the “gravitational” term grows as $\approx 1/(r-2m)$. Since this latter term is positive and will prevail (as $r \rightarrow 2m$) over the two other force terms, we are lead unavoidably to a catastrophic collapse ($\omega_{x,t} < 0$), independently on the equation of state of the configuration.

Let us now turn back to (63) and (64), to infer what happens in the non-spherical case (i.e. $\varepsilon \neq 0$), even though $\varepsilon \ll 1$. Then neglecting higher order terms on ε , and being careful with metric functions terms (some of which may tend to zero as the object becomes more and more compact) we obtain from (63)

$$\begin{aligned} - f^{-2\varepsilon} [\omega_{x,t}(\rho + p_{xx})] &= f \Delta^{1/2} \Delta^{-\varepsilon} \Sigma^\varepsilon \left(\frac{f_{,r}}{f} \right) (\rho + p_{xx})(1 + \varepsilon) \\ + f \Delta^{1/2} \Delta^{-\varepsilon} \Sigma^\varepsilon &\left\{ p_{xx,r} - \frac{1}{r} [-2p_{xx} + p_{yy} + p_{zz}] - \varepsilon \frac{\Phi_{,r}}{\Phi} (p_{xx} - p_{yy}) \right\} \\ - r^{-1} f^{1-2\varepsilon} \Phi^\varepsilon &[\cot \theta p_{xy} + p_{xy,\theta}] \end{aligned} \quad (70)$$

where we have used the fact that

$$\omega_{y,t} \approx p_{xy} \approx p_{zz} - p_{yy} \approx O(\varepsilon)$$

Since in this approximation the system (63), (64) is not longer coupled (in $\omega_{x,t}$ and $\omega_{y,t}$) we shall consider only equation (70).

In order to extract more information from (70) it is necessary to specify the source under consideration. We shall use the two configurations mentioned in section 2. In both cases, f vanishes before $a = 2m$. Thus, in the Schwarzschild-like models we have $f(0) = 0$, if $2m/a = 8/9$. Since we know that any spherically symmetric static configuration with constant ρ_{ss} and locally isotropic pressure should satisfy the constraint $n \equiv 2M/a < 8/9$

we may assume that the system leaves the equilibrium for values of n close to $8/9$. Then, an inspection of (70) shows that (if $\varepsilon > 0$), for values of n approaching $8/9$, the values of $\omega_{x,t}$ should become vanishingly small. Indeed, as $n \rightarrow 8/9$, $f \rightarrow 0$ and $p_{xx} \approx 1/f$; $f_{,r}/f \approx p_{xx}$, whereas Δ and Σ remain different from zero and bounded. Then, the largest term on the right of (70) will be of order p_{xx} (or $1/f$). So, for any finite ε (no matter how small) $\omega_{x,t}$ should be of order $f^{2\varepsilon}$.

We may now use Einstein equations to elucidate that for negative values of ε the system may be unphysical. If n takes values close to the limit allowed by the model ($8/9$ for Schwarzschild-type model and $4/5$ for Adler-type model), the critical values ($f \rightarrow 0$, $p_{xx} \rightarrow 1/f$...) appear close to $r = 0$. Outside of this region the system is basically composed by an spherical incompressible fluid plus a perturbation in ε . Thus, the physical or unphysical character of the model is determined by its behaviour close to these two limits.

The energy density, in the limit $r \rightarrow 0$, for the Schwarzschild-type model and Adler-type model is given by expresions (81) and (82) respectively. From these ones, it is easy to show that if $\varepsilon < 0$, the energy density becomes negative as the system approaches to $n \rightarrow 8/9$ (Schwarzschild case) or $n \rightarrow 4/5$ (Adler case) and positive for $\varepsilon \geq 0$. Therefore, in both cases, positive energy conditions impose $\varepsilon \geq 0$, and for very compact objects (close to the limit allowed by the model) the inertial mass term substantially increases suggesting the "stalling" of the collapse, close to, but before, the maximum allowed value of n .

5 Conclusions

We have seen so far that (as expected), for high gravitational fields, important differences appear between the spherical and the non-spherical collapse. This conclusion being true even for small non-sphericity.

In the two models considered above, a factor multiplying the inertial mass term bring out those differences. Alternatively we may multiply both sides of equation (70) by $f^{2\varepsilon}$ and say that the total "force term" decreases as $f^{2\varepsilon}$. The final result being the same, namely that $\omega_{x,t} \approx f^{2\varepsilon}$. Although for the two examples considered here, we have considered $\varepsilon > 0$, it is obvious that there exist models with $\varepsilon < 0$, in which case the effective inertial mass density term may become very small, leading to highly unstable situations and to

a breakdown of the linear approximation. In fact, it is worth noticing that important differences between the two cases ($\varepsilon > 0$ and $\varepsilon < 0$) appear also in the behaviour of the exterior γ -metric ([10], [11])

Thus, on the basis of presented results we may conclude that whatever the model and the sign of ε would be, any source of Weyl metric would evolve quite differently from the corresponding spherical source, as critical values of n are considered.

In particular, in the example examined, the inevitability of collapse in the spherical case, appears to be modified by a sharp increase in the effective inertial mass density term (or a sharp decrease in the "total force" term) as n approaches its maximum allowed value. This increase makes the system more stable, hindering its departure from equilibrium.

We would like to conclude by stressing our main point: the inevitability of the catastrophic collapse for very compact objects, which appears in the spherically symmetric case, (for any regular matter configuration) is not present at least for the family of solutions considered here (e.g. is not "strictly" unavoidable).

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A The Einstein tensor

The calculation of non-vanishing components of the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (71)$$

yields for the metric (14), using Maple V with GrTensor library and checking against the results given in [14],

$$G_{rr} = -\Sigma^{1-\gamma^2} \Delta^{\gamma^2-2} f^{2-4\gamma} \times$$

$$\begin{aligned}
& \{ \\
& \frac{f,tt}{f}(1-\gamma)^2 + \frac{\Phi,tt}{2\Phi}(1-\gamma^2) + \left(\frac{\Phi,t}{2\Phi}\right)^2 (\gamma^4 - 1) + \left(\frac{f,t}{f}\right)^2 \gamma(\gamma-2)(\gamma-1)^2 \\
& + \frac{\Phi,t}{2\Phi} \frac{f,t}{f} (1-\gamma^2)(2\gamma^2 - 4\gamma + 1) \\
& \} \\
& -r^{-2}\Sigma^{1-\gamma^2}\Delta^{\gamma^2-2}f^{2-2\gamma^2}\Phi^{\gamma^2-1}\left[1+\frac{\Phi,\theta}{2\Phi}(1-\gamma^2)\cot\theta\right] \\
& + \frac{\Phi,r}{2\Phi}(1-\gamma^2)\left[\frac{f,r}{f}+\frac{1}{r}\right] + \frac{f,r}{f}\frac{1}{r}(1+\gamma^2) + \frac{1}{r^2}
\end{aligned} \tag{72}$$

$$G_{r\theta} = (1-\gamma^2)\left[\frac{\Sigma,\theta}{2\Sigma}\left(\frac{f,r}{f}+\frac{1}{r}\right) + \left(\frac{\Phi,r}{2\Phi}-\frac{f,r}{f}\right)\cot\theta\right] \tag{73}$$

$$\begin{aligned}
G_{rt} &= -\frac{f,rt}{f}(\gamma-1)^2 + \frac{\Phi,rt}{2\Phi}(\gamma^2-1) \\
&+ \left(\frac{\Delta,t}{2\Delta}(\gamma^2-2) + (1-\gamma^2)\left[\frac{\Sigma,t}{2\Sigma} + \frac{f,t}{f}\right]\right)\left[\frac{f,r}{f}(1-\gamma)^2 + \frac{\Phi,r}{2\Phi}(1-\gamma^2)\right] \\
&+ \frac{1}{r}\left(\frac{\Delta,t}{\Delta}(\gamma^2-2) + (1-\gamma^2)\left[\frac{\Sigma,t}{\Sigma} + \frac{f,t}{f}\right]\right) \\
&+ \frac{\Phi,t}{2\Phi}\left[\frac{f,r}{f}\gamma(\gamma^2-1)(\gamma-2) + \frac{\Phi,r}{2\Phi}(1-\gamma^4) + \frac{(\gamma^2-1)}{r}\right]
\end{aligned} \tag{74}$$

$$\begin{aligned}
G_{\theta\theta} &= r^2 f^{2\gamma^2-2}\Sigma^{\gamma^2-1}\Delta^{2-\gamma^2}\Phi^{1-\gamma^2} \times \\
&\{ \\
&\frac{f,rr}{f} + \left(\frac{1}{r} + \frac{f,r}{f}\right)\left[\frac{\Sigma,r}{2\Sigma}(\gamma^2-1) - \frac{\Delta,r}{2\Delta}(\gamma^2-2)\right] + \frac{f,r}{f}\left(\frac{1}{r} + (\gamma^2-1)\frac{f,r}{f}\right) \\
&\}
\end{aligned}$$

$$\begin{aligned}
& +r^2 f^{2\gamma(\gamma-2)} \Phi^{1-\gamma^2} \times \\
& \{ \\
& \frac{\Delta_{,t}}{2\Delta}(\gamma^2 - 2) \left[\frac{\Delta_{,t}}{2\Delta}(4 - \gamma^2) + \frac{\Sigma_{,t}}{\Sigma}(\gamma^2 - 1) + \frac{f_{,t}}{f}(4\gamma - 3) \right] \\
& + \frac{\Sigma_{,t}}{2\Sigma}(1 - \gamma^2) \left[\frac{\Sigma_{,t}}{2\Sigma}(1 + \gamma^2) + \frac{f_{,t}}{f}(4\gamma - 3) \right] + \left(\frac{f_{,t}}{f} \right)^2 (1 - \gamma)(5\gamma - 1) \\
& + \frac{\Delta_{,tt}}{2\Delta}(2 - \gamma^2) + \frac{\Sigma_{,tt}}{2\Sigma}(\gamma^2 - 1) + \frac{2f_{,tt}}{f}(\gamma - 1) \\
& \} \\
& + \frac{\Sigma_{,\theta}}{2\Sigma}(1 - \gamma^2) \cot \theta
\end{aligned} \tag{75}$$

$$\begin{aligned}
G_{\theta t} = & \left(\frac{\Phi_{,t}}{2\Phi} - \frac{f_{,t}}{f} \right) (1 - \gamma^2) \cot \theta + \frac{\Sigma_{,t\theta}}{2\Sigma}(\gamma^2 - 1) - \frac{\Sigma_{,\theta}}{2\Sigma}(1 - \gamma^2) \times \\
& \left[\frac{f_{,t}}{f}(1 - \gamma^2) + \frac{\Delta_{,t}}{2\Delta}(\gamma^2 - 2) - \frac{\Phi_{,t}}{2\Phi}(1 - \gamma^2) - \frac{\Sigma_{,t}}{2\Sigma}(1 + \gamma^2) \right] \tag{76}
\end{aligned}$$

$$\begin{aligned}
G_{\varphi\varphi} = & r^2 \sin^2 \theta \Delta^{2-\gamma^2} \Sigma^{\gamma^2-1} \times \\
& \{ \\
& \frac{\Phi_{,r}}{2\Phi} \left[(\gamma^2 - 1) \left(\frac{\Phi_{,r}}{2\Phi}(\gamma^2 + 1) + \frac{\Delta_{,r}}{2\Delta}(\gamma^2 - 2) - \frac{f_{,r}}{f}(2\gamma^2 - 1) - \frac{2}{r} \right) \right] \\
& - \frac{\Phi_{,r}}{2\Phi} \frac{\Sigma_{,r}}{2\Sigma}(1 + \gamma)^2(1 - \gamma)^2 + \frac{\Phi_{,rr}}{2\Phi}(1 - \gamma^2) + \frac{f_{,rr}}{f}\gamma^2 \\
& + \frac{f_{,r}}{f}\gamma^2 \left[\frac{\Delta_{,r}}{2\Delta}(2 - \gamma^2) + (\gamma^2 - 1) \left(\frac{\Sigma_{,r}}{2\Sigma} + \frac{f_{,r}}{f} \right) \right] \\
& + \frac{1}{r} \left[\frac{\Delta_{,r}}{2\Delta}(2 - \gamma^2) + \frac{\Sigma_{,r}}{2\Sigma}(\gamma^2 - 1) + \frac{f_{,r}}{f}(2\gamma^2 - 1) \right] \\
& \}
\end{aligned}$$

$$\begin{aligned}
& + r^2 \sin^2 \theta f^{2-4\gamma} \times \\
& \{ \\
& \frac{\Phi_{,t}}{2\Phi} \left[(1-\gamma^2) \left(\frac{\Phi_{,t}}{2\Phi} (1+\gamma^2) + \frac{\Delta_{,t}}{2\Delta} (2-\gamma^2) - \frac{f_{,t}}{f} (2\gamma^2 - 4\gamma + 1) \right) \right] \\
& - \frac{\Phi_{,t}}{2\Phi} \frac{\Sigma_{,t}}{2\Sigma} (1+\gamma)^2 (1-\gamma)^2 \\
& + \frac{\Delta_{,t}}{2\Delta} (\gamma^2 - 2) \left[(4-\gamma^2) \frac{\Delta_{,t}}{2\Delta} + \frac{\Sigma_{,t}}{\Sigma} (\gamma^2 - 1) \right] + \left(\frac{\Sigma_{,t}}{2\Sigma} \right)^2 (1-\gamma^4) \\
& + \frac{f_{,t}}{f} \left[\frac{f_{,t}}{f} \gamma (2-\gamma) (\gamma-1)^2 + (\gamma^2 - 4\gamma + 2) \left(\frac{\Delta_{,t}}{2\Delta} (2-\gamma^2) - \frac{\Sigma_{,t}}{2\Sigma} (1-\gamma^2) \right) \right] \\
& - \frac{f_{,tt}}{f} (1-\gamma)^2 + \frac{\Delta_{,tt}}{2\Delta} (2-\gamma^2) + (\gamma^2 - 1) \left[\frac{\Sigma_{,tt}}{2\Sigma} + \frac{\Phi_{,tt}}{2\Phi} \right] \\
& \} \\
& + \sin^2 \theta f^{2(1-\gamma^2)} \Phi^{\gamma^2-1} \times \\
& \left[\frac{\Sigma_{,\theta\theta}}{2\Sigma} (1-\gamma^2) + \frac{\Sigma_{,\theta}}{2\Sigma} \left(\frac{\Sigma_{,\theta}}{2\Sigma} (\gamma^4 - 1) - \frac{\Phi_{,\theta}}{2\Phi} (1+\gamma)^2 (1-\gamma^2) \right) \right] \tag{77}
\end{aligned}$$

$$\begin{aligned}
G_{tt} = & \frac{\Sigma_{,t}}{2\Sigma} (1-\gamma)^2 \left[\frac{\Phi_{,t}}{2\Phi} (1+\gamma)^2 + \frac{f_{,t}}{f} (1-\gamma^2) \right] \\
& + \frac{\Delta_{,t}}{2\Delta} (\gamma^2 - 2) \left[\frac{\Phi_{,t}}{2\Phi} (1-\gamma^2) + \frac{f_{,t}}{f} (1-\gamma)^2 \right] \\
& + \frac{f_{,t}}{f} (1-\gamma)^2 \left[\frac{\Phi_{,t}}{2\Phi} (2+2\gamma) + \frac{f_{,t}}{f} (1-2\gamma) \right] + f^{4\gamma-2} \Delta^{2-\gamma^2} \Sigma^{\gamma^2-1} \times \\
& \{ \\
& \frac{\Delta_{,r}}{2\Delta} (\gamma^2 - 2) \left[\frac{f_{,r}}{f} (1-\gamma)^2 + \frac{\Phi_{,r}}{2\Phi} (1-\gamma^2) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\Sigma_{,r}}{2\Sigma}(1-\gamma)^2 \left[\frac{f_{,r}}{f}(1-\gamma^2) + \frac{\Phi_{,r}}{2\Phi}(1+\gamma)^2 \right] \\
& + \frac{f_{,r}}{f}(1-\gamma)^2 \left[\frac{f_{,r}}{f}(1-\gamma^2) + \frac{\Phi_{,r}}{\Phi}\gamma(1+\gamma) \right] + (1-\gamma^4) \left(\frac{\Phi_{,r}}{2\Phi} \right)^2 \\
& + \frac{\Phi_{,rr}}{2\Phi}(\gamma^2-1) - \frac{f_{,rr}}{f}(1-\gamma)^2 + \frac{1}{r} \left[\frac{\Delta_{,r}}{\Delta}(\gamma^2-2) + \frac{\Sigma_{,r}}{\Sigma}(1-\gamma^2) \right] \\
& + \frac{1}{r} \left[\frac{f_{,r}}{f}(3\gamma-1)(1-\gamma) + 3\frac{\Phi_{,r}}{2\Phi}(\gamma^2-1) - \frac{1}{r} \right] \\
& \} \\
& + r^{-2} f^{2\gamma(2-\gamma)} \Phi^{\gamma^2-1} \times \\
& \{ \\
& \frac{\Sigma_{,\theta\theta}}{2\Sigma}(\gamma^2-1) + \frac{\Sigma_{,\theta}}{2\Sigma} \left[\frac{\Phi_{,\theta}}{2\Phi}(1-\gamma)^2(1+\gamma)^2 + \frac{\Sigma_{,\theta}}{2\Sigma}(1-\gamma^4) \right] \\
& + (1-\gamma^2) \left[\frac{\Phi_{,\theta}}{2\Phi} - \frac{\Sigma_{,\theta}}{2\Sigma} \right] \cot \theta + 1 \\
& \}
\end{aligned} \tag{78}$$

where a comma denote partial derivation.

B Energy density for $\gamma = 1 + \varepsilon$

Since we are interested in the effect of small deviations from spherical symmetry, we shall assume that

$$\gamma = 1 + \varepsilon, \tag{79}$$

where ε is a small constant that may be positive or negative. We restrict ourselves to first order in ε , so we shall neglect terms of order $\mathcal{O}(\varepsilon^2)$ and higher.

Then, the tt component of Einstein equations reads

$$\begin{aligned}
8\pi\rho &= f^{2\varepsilon} \Delta^{1-2\varepsilon} \Sigma^{2\varepsilon} \times \\
&\{
\end{aligned}$$

$$\begin{aligned}
& \varepsilon \frac{\Delta_{,r}}{2\Delta} \frac{\Phi_{,r}}{\Phi} - \varepsilon \left(\frac{\Phi_{,r}}{\Phi} \right)^2 + \varepsilon \frac{\Phi_{,rr}}{\Phi} \\
& + \frac{1}{r} \left((2\varepsilon - 1) \frac{\Delta_{,r}}{\Delta} - 2\varepsilon \left[\frac{\Sigma_{,r}}{\Sigma} + \frac{f_{,r}}{f} \right] + 3\varepsilon \frac{\Phi_{,r}}{\Phi} - \frac{1}{r} \right) \\
& \} \\
& + f^{-2\varepsilon} \Phi^{2\varepsilon} \times \left[\varepsilon \left(\frac{\Sigma_{,\theta\theta}}{\Sigma} - \left[\frac{\Sigma_{,\theta}}{\Sigma} \right]^2 - \cot \theta \left[\frac{\Phi_{,\theta}}{\Phi} - \frac{\Sigma_{,\theta}}{\Sigma} \right] \right) + 1 \right] \quad (80)
\end{aligned}$$

B.1 Schwarzschild-type solution

Using (15-20) into (80) it is easy to show that, for $r \rightarrow 0$, the energy density can be expressed as

$$\rho = \frac{m}{2\pi a^3} f^{2\varepsilon} \left[\frac{3}{2} + \varepsilon \left(1 + \frac{1}{3\sqrt{1 - \frac{2m}{a}} - 1} \right) \right] \quad (81)$$

B.2 Adler-type solution

In this case the energy density for $r \rightarrow 0$ is given, by means of (21-26) and (80), by expression

$$\rho = \frac{3m}{4\pi a^3 (1 - 2m/a)^{1/3}} \left(\frac{1 - 5m/2a}{\sqrt{1 - 2m/a}} \right)^{2\varepsilon - 2/3} \left[(1 - m/a)^{2/3} + \frac{\varepsilon}{(1 - 5m/2a)^{1/3}} \right] \quad (82)$$

C Conservation equations

The left hand side of conservation equations

$$T_{\nu;\mu}^{\mu} = 0$$

are

$$T_{t;\mu}^{\mu} = \Gamma^2 \times \{$$

$$\begin{aligned}
& \left[\frac{f_{,t}}{f} (\gamma - 1) (\gamma - 2) + \frac{\Delta_{,t}}{2\Delta} (\gamma^2 - 2) + \left(\frac{\Phi_{,t}}{2\Phi} + \frac{\Sigma_{,t}}{2\Sigma} \right) (1 - \gamma^2) + \frac{2\Gamma_{,t}}{\Gamma} \right] \times \\
& (\rho + p_{xx} w_x^2 + p_{yy} w_y^2 + 2p_{xy} w_x w_y) \\
& + \rho_{,t} + p_{xx,t} w_x^2 + 2p_{xx} w_x w_{x,t} + p_{yy,t} w_y^2 \\
& + 2p_{yy} w_y w_{y,t} + 2p_{xy,t} w_x w_y + 2p_{xy} (w_x w_{y,t} + w_{x,t} w_y) \\
& \} \\
& + f^{2\gamma-1} \Delta^{1-\gamma^2/2} \Sigma^{(\gamma^2-1)/2} \Gamma \times \\
& \{ \\
& \left[\frac{f_{,r}}{f} (\gamma^2 + 1) + \frac{\Phi_{,r}}{2\Phi} (1 - \gamma^2) + \frac{\Gamma_{,r}}{\Gamma} + \frac{2}{r} \right] \times \\
& (\Gamma w_x \rho + p_{xx} w_x \Lambda_x + p_{yy} w_y \Lambda + p_{xy} [w_x \Lambda + w_y \Lambda_x]) \\
& + \Gamma_{,r} w_x \rho + \Gamma w_{x,r} \rho + \Gamma w_x \rho_{,r} + p_{xx,r} w_x \Lambda_x \\
& + p_{yy,r} w_y \Lambda + p_{xy,r} [w_x \Lambda + w_y \Lambda_x] \\
& + w_{x,r} [p_{xx} \Lambda_x + p_{xy} \Lambda] + w_{y,r} [p_{yy} \Lambda + p_{xy} \Lambda_x] \\
& + \Lambda_{x,r} [p_{xx} w_x + p_{xy} w_y] + \Lambda_{,r} [p_{yy} w_y + p_{xy} w_x] \\
& \} \\
& + r^{-1} f^{2\gamma-\gamma^2} \Phi^{(\gamma^2-1)/2} \Gamma \times \\
& \{ \\
& \left[\frac{\Sigma_{,\theta}}{2\Sigma} (1 - \gamma^2) + \frac{\Gamma_{,\theta}}{\Gamma} + \cot \theta \right] \times \\
& (\Gamma w_y \rho + p_{xx} w_x \Lambda + p_{yy} w_y \Lambda_y + p_{xy} [w_y \Lambda + w_x \Lambda_y]) \\
& + \Gamma_{,\theta} w_y \rho + w_{y,\theta} (\rho \Gamma + p_{yy} \Lambda_y + p_{xy} \Lambda) + w_{x,\theta} (p_{xx} \Lambda + p_{xy} \Lambda_y)
\end{aligned}$$

$$\begin{aligned}
& +\Lambda_{,\theta} (p_{xx}w_x + p_{xy}w_y) + \Lambda_{y,\theta} (p_{yy}w_y + p_{xy}w_x) \\
& +\rho_{,\theta}\Gamma w_y + p_{xx,\theta}w_x\Lambda + p_{yy,\theta}w_y\Lambda_y + p_{xy,\theta} [w_y\Lambda + w_x\Lambda_y] \\
& \} \\
& + \left[\frac{f_{,t}}{f} \gamma (\gamma - 1) + \frac{\Phi_{,t}}{2\Phi} (1 - \gamma^2) \right] \times \\
& \left(\Gamma^2 w_y^2 \rho + p_{xx} \Lambda^2 + p_{yy} \Lambda_y^2 + 2p_{xy} \Lambda \Lambda_y \right) \\
& + \frac{f_{,t}}{f} (1 - \gamma) p_{zz} \\
& + \left[\frac{f_{,t}}{f} (1 - \gamma) + \frac{\Delta_{,t}}{2\Delta} (\gamma^2 - 2) + \frac{\Sigma_{,t}}{2\Sigma} (1 - \gamma^2) \right] \times \\
& \left(\Gamma^2 w_x^2 \rho + p_{xx} \Lambda_x^2 + p_{yy} \Lambda^2 + 2p_{xy} \Lambda \Lambda_x \right) \tag{83}
\end{aligned}$$

$$\begin{aligned}
T_{r;\mu}^\mu &= -f^{1-2\gamma} \Delta^{-1+\gamma^2/2} \Sigma^{(1-\gamma^2)/2} \Gamma \times \\
& \{ \\
& \left[\frac{f_{,t}}{f} (\gamma - 1) (\gamma - 3) + \frac{\Delta_{,t}}{\Delta} (\gamma^2 - 2) + \left(\frac{\Phi_{,t}}{2\Phi} + \frac{\Sigma_{,t}}{\Sigma} \right) (1 - \gamma^2) + \frac{\Gamma_{,t}}{\Gamma} \right] \times \\
& (\Gamma w_x \rho + p_{xx} w_x \Lambda_x + p_{yy} w_y \Lambda + p_{xy} [w_x \Lambda + w_y \Lambda_x]) \\
& + \Gamma w_x \rho_{,t} + \Gamma_{,t} w_x \rho + p_{xx,t} w_x \Lambda_x + p_{yy,t} w_y \Lambda + p_{xy,t} [w_x \Lambda + w_y \Lambda_x] \\
& + w_{x,t} (\Gamma \rho + p_{xx} \Lambda_x + p_{xy} \Lambda) + w_{y,t} (p_{yy} \Lambda + p_{xy} \Lambda_x) \\
& + \Lambda_{x,t} (p_{xy} w_y + p_{xx} w_x) + \Lambda_{,t} (p_{yy} w_y + p_{xy} w_x) \\
& \} \\
& - \left[\frac{f_{,r}}{f} (\gamma^2 - \gamma + 1) + \frac{\Phi_{,r}}{2\Phi} (1 - \gamma^2) + \frac{2}{r} \right] \times
\end{aligned}$$

$$\begin{aligned}
& \left(\Gamma^2 w_x^2 \rho + p_{xx} \Lambda_x^2 + p_{yy} \Lambda^2 + 2p_{xy} \Lambda \Lambda_x \right) \\
& - \left[2\Gamma \Gamma_{,r} w_x^2 \rho + 2\Gamma^2 w_x w_{x,r} \rho + \Gamma^2 w_x^2 \rho_{,r} + p_{xx,r} \Lambda_x^2 + p_{yy,r} \Lambda^2 + 2p_{xy,r} \Lambda \Lambda_x \right] \\
& - 2\Lambda_{x,r} [p_{xx} \Lambda_x + p_{xy} \Lambda] - 2\Lambda_{,r} [p_{xy} \Lambda_x + p_{yy} \Lambda] \\
& - r^{-1} f^{1-\gamma^2} \Delta^{-1+\gamma^2/2} \Phi^{(\gamma^2-1)/2} \Sigma^{(1-\gamma^2)/2} \times \\
& \{ \\
& \left[\frac{\Sigma_{,\theta}}{\Sigma} (1 - \gamma^2) + \cot \theta \right] \times \\
& \left(\Gamma^2 w_x w_y \rho + p_{xx} \Lambda_x \Lambda + p_{yy} \Lambda_y \Lambda + p_{xy} [\Lambda^2 + \Lambda_x \Lambda_y] \right) \\
& + \Gamma^2 w_x w_y \rho_{,\theta} + p_{xx,\theta} \Lambda_x \Lambda + p_{yy,\theta} \Lambda_y \Lambda + p_{xy,\theta} [\Lambda^2 + \Lambda_x \Lambda_y] \\
& + 2\Gamma \Gamma_{,\theta} w_x w_y \rho + \Gamma^2 w_{x,\theta} w_y \rho + \Gamma^2 w_x w_{y,\theta} \rho + \Lambda_{x,\theta} [p_{xx} \Lambda + p_{xy} \Lambda_y] \\
& + \Lambda_{y,\theta} [p_{yy} \Lambda + p_{xy} \Lambda_x] + \Lambda_{,\theta} [2p_{xy} \Lambda + p_{xx} \Lambda_x + p_{yy} \Lambda_y] \\
& \} \\
& - \frac{f_{,r}}{f} \gamma \Gamma^2 \left(\rho + p_{xx} w_x^2 + p_{yy} w_y^2 + 2p_{xy} w_x w_y \right) \\
& + \left[\frac{1}{r} + \frac{f_{,r}}{f} (1 - \gamma) \right] p_{zz} \\
& + \left[\frac{1}{r} + \frac{f_{,r}}{f} \gamma (\gamma - 1) + \frac{\Phi_{,r}}{2\Phi} (1 - \gamma^2) \right] \times \\
& \left(\Gamma^2 w_y^2 \rho + p_{xx} \Lambda^2 + p_{yy} \Lambda_y^2 + 2p_{xy} \Lambda \Lambda_y \right)
\end{aligned} \tag{84}$$

$$\begin{aligned}
T_{\theta;\mu}^\mu &= -r f^{\gamma^2-2\gamma} \Phi^{(1-\gamma^2)/2} \Gamma \times \\
&\{
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{2f_{,t}}{f} (\gamma - 1)^2 + \frac{\Delta_{,t}}{2\Delta} (\gamma^2 - 2) + \left(\frac{\Sigma_{,t}}{2\Sigma} + \frac{\Phi_{,t}}{\Phi} \right) (1 - \gamma^2) + \frac{\Gamma_{,t}}{\Gamma} \right] \times \\
& (\Gamma w_y \rho + p_{xx} w_x \Lambda + p_{yy} w_y \Lambda_y + p_{xy} [w_y \Lambda + w_x \Lambda_y]) \\
& + \Gamma w_y \rho_{,t} + p_{xx,t} w_x \Lambda + p_{yy,t} w_y \Lambda_y + p_{xy,t} [w_y \Lambda + w_x \Lambda_y] \\
& + \Gamma_{,t} w_y \rho + \Lambda_{,t} [p_{xx} w_x + p_{xy} w_y] + \Lambda_{y,t} [p_{yy} w_y + p_{xy} w_x] \\
& + w_{y,t} [p_{yy} \Lambda_y + p_{xy} \Lambda + \Gamma \rho] + w_{x,t} [p_{xx} \Lambda + p_{xy} \Lambda_y] \\
& \} \\
& - r f^{\gamma^2-1} \Delta^{1-\gamma^2/2} \Phi^{(1-\gamma^2)/2} \Sigma^{(\gamma^2-1)/2} \times \\
& \{ \\
& \left[\frac{f_{,r}}{f} (2\gamma^2 - 2\gamma + 1) + \frac{\Phi_{,r}}{\Phi} (1 - \gamma^2) + \frac{3}{r} \right] \times \\
& (\Gamma^2 w_x w_y \rho + p_{xx} \Lambda_x \Lambda + p_{yy} \Lambda_y \Lambda + p_{xy} [\Lambda^2 + \Lambda_x \Lambda_y]) \\
& + \Gamma^2 w_x w_y \rho_{,r} + p_{xx,r} \Lambda_x \Lambda + p_{yy,r} \Lambda_y \Lambda + p_{xy,r} [\Lambda^2 + \Lambda_x \Lambda_y] \\
& + 2\Gamma \Gamma_{,r} w_x w_y \rho + \Gamma^2 w_{x,r} w_y \rho + \Gamma^2 w_x w_{y,r} \rho \\
& + \Lambda_{,r} [p_{xx} \Lambda_x + p_{yy} \Lambda_y + 2p_{xy} \Lambda] \\
& + \Lambda_{y,r} [p_{yy} \Lambda + p_{xy} \Lambda_x] + \Lambda_{x,r} [p_{xx} \Lambda + p_{xy} \Lambda_y] \\
& \} \\
& - \left[\frac{\Sigma_{,\theta}}{2\Sigma} (1 - \gamma^2) + \cot \theta \right] (\Gamma^2 w_y^2 \rho + p_{xx} \Lambda^2 + p_{yy} \Lambda_y^2 + 2p_{xy} \Lambda \Lambda_y) \\
& + p_{zz} \cot \theta
\end{aligned}$$

$$\begin{aligned}
& - \left[\Gamma^2 w_y^2 \rho_{,\theta} + p_{xx,\theta} \Lambda^2 + p_{yy,\theta} \Lambda_y^2 + 2p_{xy,\theta} \Lambda \Lambda_y \right] \\
& - \left[2\Gamma \Gamma_{,\theta} w_y^2 \rho + 2\Gamma^2 w_y w_{y,\theta} \rho \right] \\
& - 2\Lambda_{y,\theta} [p_{yy} \Lambda_y + p_{xy} \Lambda] - 2\Lambda_{,\theta} [p_{xx} \Lambda + p_{xy} \Lambda_y] \\
& + \frac{\Sigma_{,\theta}}{2\Sigma} (1 - \gamma^2) \left(\Gamma^2 w_x^2 \rho + p_{xx} \Lambda_x^2 + p_{yy} \Lambda^2 + 2p_{xy} \Lambda \Lambda_x \right) \tag{85}
\end{aligned}$$

$$T^\mu_{\varphi;\mu} = -p_{zz,\varphi} \tag{86}$$

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